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Relativistic Bound States

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The Hamiltonian for Dirac's second-order equation depends nonlinearly on the potential V and the energy E. For this reason the magnetic contribution to the Hamiltonian for s-waves, which has a short range, is attractive for a repulsive Coulomb potential ($V > 0$) and repulsive for an attractive Coulomb potential ($V < 0$). Previous studies are confined to the latter case, where strong net attraction near a high-Z nucleus accelerates electrons to velocities close to the speed of light.

The Hamiltonian is linear in the product EV/mc^2. Usually solutions are found in the regime $E = mc^2 + \varepsilon$, where except for high Z, $|\varepsilon| << mc^2$. Here we show that for $V > 0$ the attractive magnetic term and the repulsive linear term combine to support a bound state at $E = 0.5\ mc^2$ corresponding to a binding energy $E_b = -\varepsilon = 0.5\ mc^2$.

I. Introduction

Dirac's equation is used in atomic and optical physics to describe the relativistic motion of an electron in the vicinity of a high-Z nucleus [1-3]. Of what conceivable interest could Dirac's equation have for a posively charged particle which under all known laboratory conditions slows down near the nucleus?

The second-order form of the Dirac equation has an s-wave magnetic contribution - the magnetic contributions of course depend on the particle's orbital angular momentum - of the form,

$$V_{mag}\psi = \frac{\hbar^2}{m}\frac{V'}{E - V + mc^2}\left(\frac{\psi}{r} - \psi'\right) \quad , \tag{1}$$

where the primes denote radial derivatives. For a repulsive Coulomb potential, $V = e^2/r$, the part of V_{mag} which multiples the wave function is attractive for $r > e^2/(E + mc^2)$ and repulsive for $r < e^2/(E + mc^2)$. Both solutions are regular at the point $r = e^2/(E + mc^2)$ where the denominator vanishes, and in the usual way one solution is regular and one is irregular at $r = 0$.

Here we report the existence of a bound state due to the attractive nature of V_{mag} near $E = 0$, where the part of the Hamiltonian linear in V, which is repulsive, vanishes. A value of E equal to zero corresponds to a binding energy of mc^2.

Although the Klein-Gordon equation is known to support a bound state with binding energy of order mc^2 for an attractive Coulomb potential with unit-strength point source at the origin, $V = - e^2/r$, [4-5], strong binding due to weak forces by the Dirac equation to our knowledge has not been previously reported.

The question naturally arises if such states exist in the physical world. The equations of general relativity have multiple solutions, only one of which discovered by Lemaitre [6] for an expanding universe, has been confirmed experimentally. I regard these weak-force strong-binding solutions of relativistic quantum theory as the counterpart in special relativity of general relativity solutions which are known but not confirmed experimentally, and this paper is intended to stimulate experimental and further theoretical interest.

If nothing else the mathematical existence of these states teaches us that the postulate of the existence of a strong force in nature responsible for nuclear binding, while it may be supported experimentally, is nevertheless rooted in our experience of particle binding in nonrelativistic quantum mechanics, which requires a strong potential

II. Relativistic bound states

Dirac's reduced radial equations are,

$$G_\kappa' + \frac{\kappa}{r} G_\kappa = \frac{1}{\hbar c}(E_S - V) F_{-\kappa} \tag{2a}$$

$$F_{-\kappa}' - \frac{\kappa}{r} F_{-\kappa} = -\frac{1}{\hbar c}(E_L - V) G_\kappa \; , \tag{2b}$$

where I have used the compact notation $E_S = E + mc^2$ and $E_L = E - mc^2$. Bound and continuum states are found in the regimes $E_S E_L < 0$ and $E_S E_L > 0$ respectively. In this notation $- E_L$ for $E_L < 0$ is the binding energy and E_S is associated with the particle magnetic moment of magnitude $e\hbar c/E_S$.

Eqs. (2) are solved for $\kappa = -1$. I first solve the second-order or Schroedinger form of the equation for G_{-1} by elimination of the equation of F_1. The second-order form is,

$$-G''_{-1} - \frac{V'\left(\frac{d}{dr} - \frac{1}{r}\right) G_{-1}}{E_S - V} - \frac{1}{\hbar^2 c^2}(E_L - V)(E_S - V)\, G_{-1} = 0 \quad . \tag{3}$$

Then knowning G_{-1} I find F_1 by writing the solution of Eq. (2b) as,

$$F_1 = -\frac{1}{\hbar c r}\int_0^r dr' r'(E_L - V)\, G_{-1} \quad , \tag{4}$$

and performing the quadrature.

This elimination isolates the singular, attractive magnetic potential in the equation for G_{-1} such that one may look for short-range binding. This is why I made the comparison earlier of the solutions of relativistic quantum theory with those of general relativity. Such studies appear not to have been made in Dirac theory, whose known solutions for the case of a scalar potential V are confined to solutions which reduce to Schroedinger's solutions in the limit $|V| << mc^2$. This limit does not obtain in the regime of the present study.

The potential V occurs in a highly nonlinear manner in the second-order form of the relativistic equation [Eq. (3)]. With reference to the second term on the left-hand-side of Eq. (3)

(magnetic contribution) the important points are as follows:

(1) the usual nonrelativistic limit is found when $| V | << | E_S |$, which occurs away from the origin for a potential with a r^{-1} singularity and for $E_S \approx 2mc^2$. This condition is satisfied for the Dirac definition $E_S = E + mc^2$ in the regime $E \approx mc^2$. (But even for the uranium atom the binding energy of a 1s electron is only about 26% of mc^2.) (2) The magnetic contribution has singular points at the origin and at the radial value displaced from the origin at which $E_S - V = 0$. With reference to Fig. 1 the magnetic term is strongly attractive for $| V | < | E_S |$ (for r outside the displaced singular point) and strongly repulsive for $| V | > | E_S |$ (for r inside the displaced singular point). For $V < 0$ the only singular point is the usual one at the origin and the magnetic term is repulsive for all r. Analysis shows that both solutions of Eq. (3) for $V > 0$ are regular at the displaced singular point. The singularity at the origin is the usual one for the Dirac $\kappa = -1$ radial equation with regular and irregular solutions going as r and r^{-1} respectively.

Fig. 1 shows a blow-up of the effective potential near the singularity displaced from the origin where $E_S - V = 0$. The effective potential is defined by $V_{eff} = \frac{\hbar^2 c^2}{E_S} U_{eff}$ where with reference to Eq. (3),

$$U_{eff} = \frac{1}{E_S - V}\frac{V'}{r} - \frac{1}{\hbar^2 c^2}[(E_L - V)(E_S - V) - E_L E_S] \quad . \tag{5}$$

Fig. 2 shows V_{eff} over an extended range of r and the radial eigenfunctions given by Eqs.(3) and (4). Eq. (3) is solved by successively integrating forward from the origin into the displaced singularity and backward from large r into the displaced singularity until the wronskian of forward and backward solutions is zero to acceptable accuracy, which in our calculation is taken to be better than 3 parts per 10^5. Eq. (4) for F_1 is then solved by back substitution of G_{-1} from Eq. (3) and quadrature.

We found that the wronskian changed sign once from positive to negative over the range of trial values of E_L from -1.5 mc^2 to -0.5 mc^2. The zero of the wronskian to the accuracy reported above was found to occur for $E_L = -E_{binding} = -0.50000\ mc^2$.

III. Summary and conclusions

We have reported the existence of a bound state with binding energy near mc^2 supported by the Dirac equation for a point-source Coulomb model. The binding is due to a net short-range magnetic attraction which dominates the Hamiltonian when the long-range Coulomb barrier vanishes at E = 0. This state represents a genre of binding - that of strong binding due to the weak electromagnetic force whose basis is the relativistic nature of the motion - which is still unobserved in the laboratory or at least, if observed, is explained by postulating the existence of a strong force in nature. Curiously it mimicks the short-range potential in nuclear physics, which lurks behind a repulsive Coulomb barrier.

Transitions to the state from higher energy states might be expected to be highly unlikely due to the Coulomb barrier which exists in the normal region of the spectrum for $E \approx mc^2$. Further theoretical work to investigate if Levenson's theorem could be applied to the normal spectral states in order to infer the existence of an "extra" state would be highly desirable. Finally it would be desirable to design experiments which would confirm or deny the existence of the state and the genre of binding which it represents.

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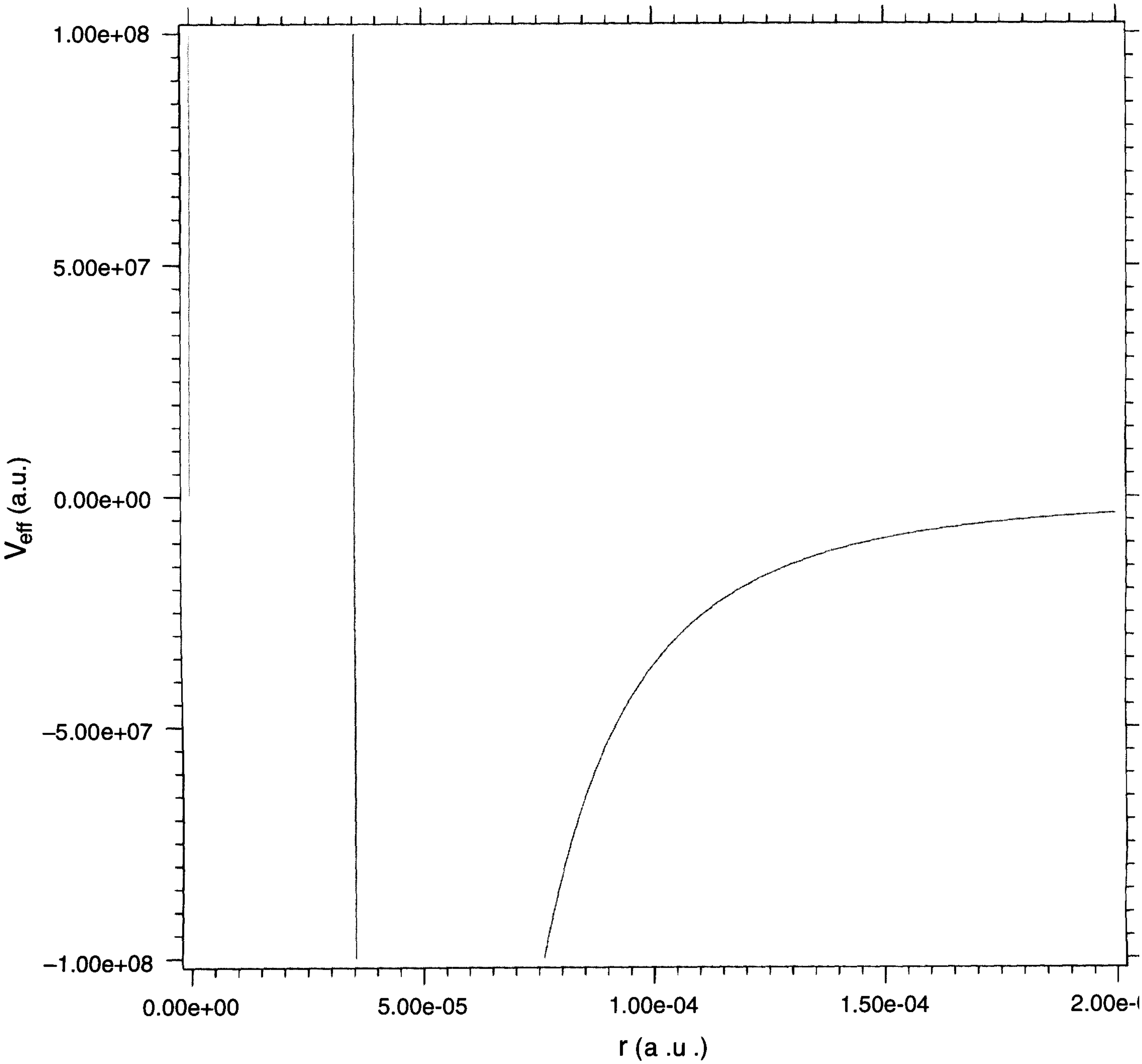
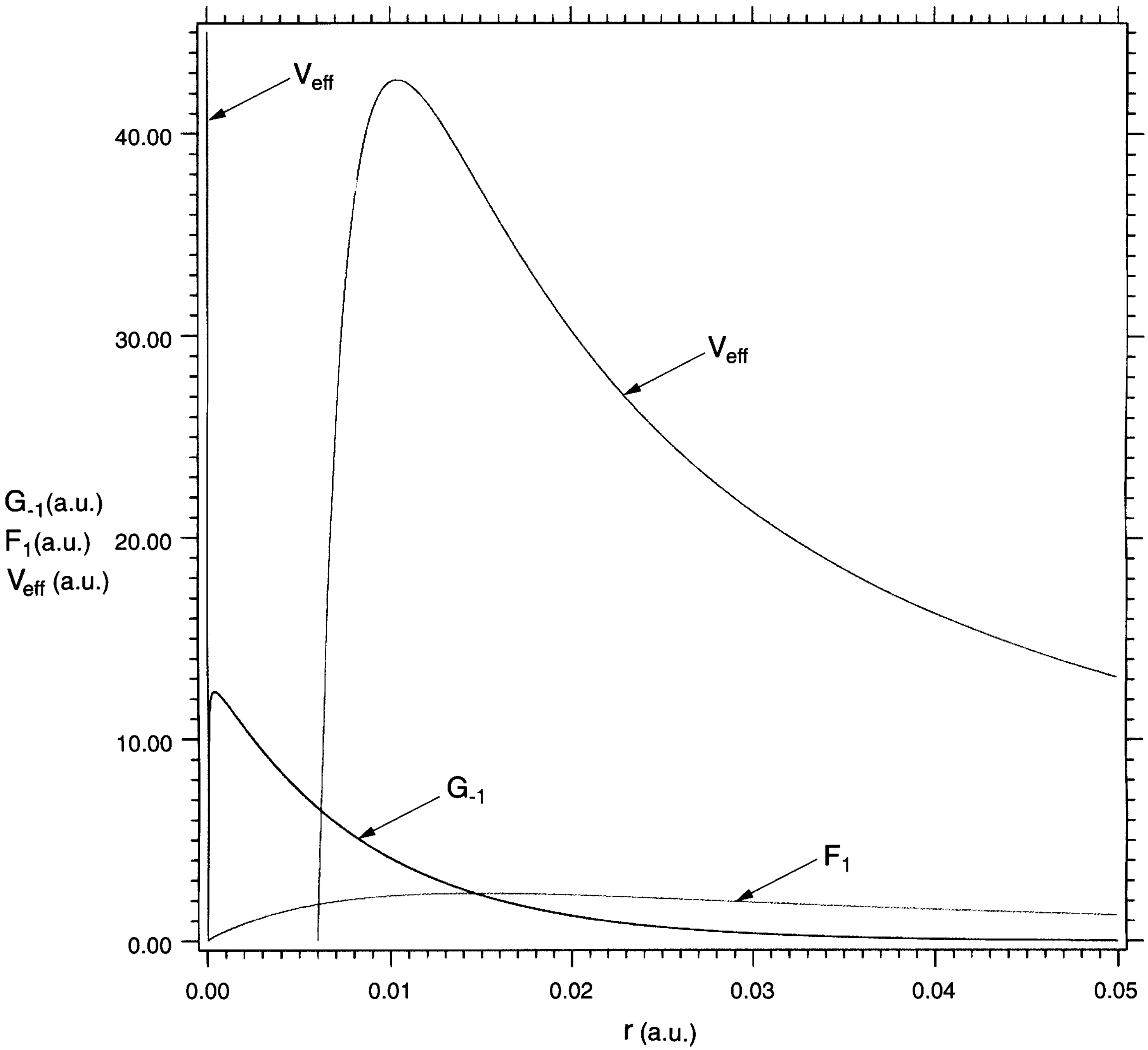
Figure captions

Fig. 1. Blow up of V_{eff} [Eq. (5)] versus r near the singular point displaced from the origin where $E_S - V = 0$. For $E_S = mc^2$ and $V = e^2/r$, the displaced singular point occurs at $r = e^2/mc^2$.

Fig. 2. V_{eff} [Eq. (5)] and normalized eigenfunctions G_{-1} [Eq. (3)] and F_1 [Eq. (4)] versus r.



1.00e+08
5.00e+07
0.00e+00
−5.00e+07
−1.00e+08
V_{eff} (a.u.)
0.00e+00
5.00e-05
1.00e-04
1.50e-04
r (a .u .)



V_{eff}
V_{eff}
G_{-1}
F_1
G_{-1}(a.u.)
F_1(a.u.)
V_{eff} (a.u.)
40.00
30.00
20.00
10.00
0.00
0.00
0.01
0.02
0.03
0.04
0.05
r (a.u.)